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THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS IN THE PROBLEM OF THE HELMHOLTZ ACOUSTIC RESONATOR†

R. R. GADYL'SHIN

Ufa

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A Helmholtz resonator of fairly arbitrary form is considered. The asymptotic form with respect to a small parameter (the linear dimensions of the aperture) is constructed for the scattered field.

1. FORMULATION OF THE PROBLEM

SUPPOSE that a bounded region $\Omega \subset R^3$ has a fairly smooth boundary Γ_0, while Γ_ε is obtained from Γ_0 by cutting out an aperture ω_ε with linear dimensions of the order of $0<\varepsilon\ll 1$ (a Helmholtz acoustic resonator). We will assume that the space is filled with a uniform and isotropic liquid or gas.

If the value of the normal component of the potential velocity $\mathbf{v}_\varepsilon = \operatorname{grad} u_\varepsilon$ is specified on the surface

$$\frac{\partial u_\varepsilon}{\partial \mathbf{n}} = f \text{ on } \Gamma_\varepsilon \tag{1.1}$$

the potential u_ε is a solution of the Helmholtz equation which satisfies the Sommerfeld radiation condition

$$(\Delta+k^2)u_\varepsilon=0 \quad x\in R^3\backslash\Gamma_\varepsilon \tag{1.2}$$

$$\frac{\partial u_\varepsilon}{\partial r} - iku_\varepsilon = o(r^{-1}) \quad (r\to\infty) \tag{1.3}$$

and the Meixner condition at the edge of the surface Γ_ε. Here and everywhere henceforth $\mathbf{n}$ is the external normal to Ω, $x=(x_1, x_2, x_3)$, $r=|x|$.

The problem of finding the scattered field u_ε which occurs when a plane wave $u^{\text{in}} = A_0 e^{i(\mathbf{kx})}$ is reflected from an ideally rigid surface Γ_ε, $k=|\mathbf{k}|$ can also be reduced to solving the boundary-value problem (1.1)–(1.3). In this case we must put $f=-\partial u^{\text{in}}/\partial\mathbf{n}$ in (1.1).

Resonance phenomena in (1.1)–(1.3) arise for k close to k_0 where k_0^2 is the eigenvalue of the Neumann problem for the Laplace operator in the region Ω. In particular, when a plane wave is incident, the field reflected from Γ_ε differs from the field reflected from Γ_0 by a quantity of the order of $O(1)$ as $\varepsilon\to 0$. These phenomena were investigated by Helmholtz and Rayleigh [1] for a sphere with a small aperture and is still being investigated at the present time (see, for example, [2] and the review of the literature contained there). In this paper we consider a resonator of fairly arbitrary form.

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2. FUNDAMENTAL RESULTS

The resonance phenomena can be explained as follows [3–5]: Green's function $G_\varepsilon(x, y, k)$ of boundary-value problem (1.1)–(1.3) admits of an analytical extension with respect to k into the complex plane and has poles τ_ε in the half-plane $\operatorname{Im} k<0$. Some of these poles, in the limit as $\varepsilon \to 0$, acquire real values k_0, where k_0^2 are the eigenvalues of the Neumann problem in Ω. Also, if $k_0^2 \neq 0$ is a simple eigenvalue (it is this case which we shall be considering), there is a unique pole of the first order $\tau_\varepsilon \to k_0$ as $\varepsilon \to 0$. The presence of this pole also explains the resonance phenomena for real k close to k_0. The expression for the solution of (1.1)–(1.3) in terms of Green's function gives the following representation for it:

$$u_\varepsilon(x, k) = \frac{\psi_\varepsilon(x)}{\tau_\varepsilon^2 - k^2} \int\limits_{\Gamma_\varepsilon} \{\psi_\varepsilon(y)\} f(y)\, dy + U_\varepsilon(x, k) \tag{2.1}$$

where $2\pi i \psi_\varepsilon(x)\psi_\varepsilon(y)$ is the residue of the function $G_\varepsilon(x, y, k)$ at the pole $k = \tau_\varepsilon$ and the quasi-eigenfunction $\psi_\varepsilon(x)$ is the solution of the boundary-value problem (1.1), (1.2) when $k = \tau_\varepsilon$,

$$\{\psi_\varepsilon(x)\} = \lim_{y\to x} \psi_\varepsilon(y) - \lim_{z\to x} \psi_\varepsilon(z) \quad (x \in \Gamma_0,\ y \in \Omega,\ z \notin \bar{\Omega})$$

We will denote by $\psi_0(x)$ the eigenfunction of the Neumann problem for the Laplace operator in Ω, corresponding to the eigenvalue k_0^2, normalized in $L_2(\Omega)$ and extended to zero outside $\bar{\Omega}$. We will also denote by $U_0(x; k)$ the solution of the boundary-value problem

$$(\Delta + k^2) U_0 = 0 \quad x \notin \bar{\Omega},\ \frac{\partial U_0}{\partial n} = f \quad x \in \Gamma_0 \tag{2.2}$$

which satisfies the radiation condition (1.3).

It was shown† that $\psi_\varepsilon \to \psi_0$ when $\varepsilon \to 0$ in $W_2^1(\Omega)$ and $W_2^1(K \backslash \bar{\Omega})$

$$\int\limits_{\Gamma_\varepsilon} \{\psi_\varepsilon\} f\, dx \to \int\limits_{\Gamma_0} \psi_0 f\, dx \tag{2.3}$$

when $\varepsilon \to 0$, while $U_\varepsilon \to U_0$ and $W_2^1(K \backslash \bar{\Omega})$ uniformly with respect to k close to k_0, and is bounded in $W_2^1(\Omega)$ uniformly with respect to ε and k close to zero and k_0, respectively and K is an arbitrary compactum in R^3.

It is obvious that for k close to k_0 the first term in (2.1) makes the main contribution to the solution u_ε. For a more accurate estimate of this contribution it is necessary to know the behaviour of τ_ε and $\psi_\varepsilon(x)$ as $\varepsilon \to 0$. The asymptotic form of these quantities is constructed by matching the asymptotic expansions [6, 7] and is similar in its approach to the construction of the asymptotic forms of the eigenvalues of elliptic boundary-value problems in singularly perturbed regions [8, 9]. We will carry out this construction below.

Assuming that Ω in the neighbourhood of the origin of coordinates is identical with the half-space $x_3>0$, ω is a two-dimensional region with a smooth boundary in the plane $x_3 = 0$, and $\omega_\varepsilon = \{x; x\varepsilon^{-1} \in \omega\}$, the asymptotic form of the pole τ_ε has the form

$$\tau_\varepsilon = k_0 + \sum_{j=1}^{\infty} \varepsilon^j \tau_j$$

$$\tau_1 = \pi\psi_0^2(0) c_\omega/(2k_0), \quad \operatorname{Im} \tau_2 = -\sigma(\pi\psi_0(0) c_\omega)^2/2 \tag{2.4}$$

where c_ω is the capacity of the "plate" ω [10] and σ is the transverse cross-section [11] of Green's function $G(x, y, k_0)$ of the Neumann problem for the Helmholtz operator outside Ω when $y = 0$.

We have the following expansion for $\psi_\varepsilon(x)$:

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$$\psi_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j \psi_j(x),\ x \in \Omega \setminus S(\varepsilon^{1/3}) \tag{2.5}$$

$$\psi_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j v_j(x/\varepsilon),\ x \in S(2\varepsilon^{1/2}) \tag{2.6}$$

$$\psi_\varepsilon(x) = \sum_{j=1}^{\infty} \varepsilon^j P_j(D_y) G(x, 0, \tau_\varepsilon), \quad x \notin \Omega \cup S(\varepsilon^{1/3}), \quad P_0 = \pi\psi_0(0)\, c_\omega \tag{2.7}$$

uniform in $W_2^1(\Omega)$ and $W_2^1(K\setminus\bar{\Omega})$ for any compactum $K \subset R^3$. Here $s(t)$ is a sphere of radius t with centre at the origin of coordinates, and $P_j(D_y)$ are differential polynomials of the jth order in the variable y with constant coefficients.

Suppose $\psi_0(0) \neq 0$. *It then follows from* (2.1) *and* (2.4) *that resonance* phenomena are observed to the greatest extent for real values of

$$k = k_0 + \varepsilon\tau_1 + \varepsilon^2(k_2 + o(1)) \tag{2.8}$$

If, moreover,

$$a_f = \int_{\Gamma_0} \psi_0 f\, dx \neq 0 \tag{2.9}$$

then, substituting the asymptotic expansions (2.4)–(2.7) into (2.1) and taking (2.3) into account, we obtain the following representation for the solution of boundary-value problem (1.1)–(1.3):

$$\begin{gathered} u_\varepsilon(x;\mathbf{k}) \sim A_f \varepsilon^{-2}\psi_0(x),\ x \in \Omega\setminus S(\varepsilon^{1/2});\ u_\varepsilon(x;\mathbf{k}) \sim A_f\varepsilon^{-2} v_0(x/\varepsilon) \\ x \in S(2\varepsilon^{1/2});\ u_\varepsilon(x;\mathbf{k}) \sim A_f \varepsilon^{-1} P_0 G(x, 0, k),\ x \notin \Omega \cup S(\varepsilon^{1/2}) \\ A_f = a_f(2k_0(\tau_2 - k_2))^{-1} \end{gathered} \tag{2.10}$$

which holds for any compactum $K \subset R^3$ in $W_2^1(\Omega)$, $W_2^1(K\setminus\bar{\Omega})$.

The situation is somewhat different when finding the scattered field u_ε which occurs when a plane wave u^{in} is reflected from an ideally rigid surface Γ_ε, since condition (2.9) is not satisfied.

Suppose $u_0(x;\mathbf{k})$ is the scattered field which occurs outside Ω when a plane wave $u^{\text{in}}(x;\mathbf{k})$ is reflected from Γ_0 [the solution of boundary-value problem (2.2), (1.3) for $f = -\partial u^{\text{in}}/\partial\mathbf{n}$], while $u = u_0 + u^{\text{in}}$ in $R^3\setminus\Omega$.

Integrating by parts the left-hand side of the equation

$$\int_{S(R)} u^{in}(x;\mathbf{k})(\Delta + \tau_\varepsilon^2)\psi_\varepsilon(x)\,dx = 0$$

for fairly large fixed R and taking into account the asymptotic form (2.8) of the function $\psi_\varepsilon(x)$, we obtain the following relation:

$$\begin{aligned} &\int_{\Gamma_\varepsilon} \{\psi_\varepsilon(x)\} \frac{\partial u^{\text{in}}(x;\mathbf{k})}{\partial\mathbf{n}}\, ds = \\ &= -\varepsilon P_0 u(0;\mathbf{k}_0) + O(\varepsilon(|k - k_0| + |\tau_\varepsilon - k_0|) + (k - \tau_\varepsilon)) \end{aligned} \tag{2.11}$$

We will assume that $u(0,\mathbf{k}_0) \neq 0$ while k satisfies relation (2.8). Then, substituting (2.5)–(2.8) and (2.11) into (2.1) we obtain the principal terms of the asymptotic form for the scattered field

$$\begin{gathered} u_\varepsilon(x;\mathbf{k}) \sim b\varepsilon^{-1}\psi_0(x),\ x \in \Omega\setminus S(\varepsilon^{1/2});\ u_\varepsilon(x;\mathbf{k}) \sim b\varepsilon^{-1} v_0(x/\varepsilon) \\ x \in S(2\varepsilon^{1/2});\ u_\varepsilon(x;\mathbf{k}) \sim bP_0 G(x, 0, k) + u_0(x, k_0),\ x \notin \Omega \cup S(\varepsilon^{1/2}) \\ b = u(0, \mathbf{k}_0) P_0 (2k_0(\tau_2 - k_2))^{-1} \end{gathered} \tag{2.12}$$

Representation (2.12) holds in $W_2^1(\Omega)$ and $W_2^1(K\backslash\bar{\Omega})$ for any compactum $K\subset R^3$.

3. CONSTRUCTION OF THE ASYMPTOTIC FORM OF τ_ε

We will show that relations (2.4)–(2.7) hold. The series (2.5)–(2.7) were constructed to be asymptotically matched at zero: i.e. the double series obtained from (2.5) by replacing its coefficients by their asymptotic forms at zero is identical with the double series obtained from (2.6) by replacing the coefficients by their asymptotic forms when $\rho = r\varepsilon^{-1}\to\infty$, $\xi_\varepsilon\geqslant 0$. Similarly, the series obtained from (2.7) by replacing the coefficients by their asymptotic forms at zero is identical with the series obtained from (2.6) by replacing the coefficients by their asymptotic forms when $\rho\to\infty$, $\xi_3\leqslant 0$. The series (2.4)–(2.7) cannot be constructed independently of one another. These series can only be constructed completely by comparing them.

The boundary-value problems for the coefficients of series (2.5) are obtained by substituting the series (2.4) and (2.5) into (1.1) and (1.2), by writing the equations separately for the same powers of ε and passing to the formal limit as $\varepsilon\to 0$

$$(\Delta + k_0^2)\psi_j = -2k_0\sum_{l=1}^{j-1}\tau_l\psi_{j-l}, \quad x\in\Omega, \quad \partial\psi_j/\partial\mathbf{n} = 0, \quad x\in\Gamma_0\setminus 0 \tag{3.1}$$

Similarly, in the variables $\xi = x\varepsilon^{-1}$ we obtain boundary-value problems for the coefficients of the series (2.6). In particular,

$$\Delta_\xi v_0=0 \quad \xi\notin\bar{\gamma}, \quad \partial v_3/\partial\xi_3=0 \quad \xi\in\gamma \tag{3.2}$$

where $\gamma = R^2\backslash\bar{\omega}$, R^2 is the plane $\xi_3 = 0$.

Green's function $G(x, y, k)$ is continuous for real k and admits of an analytical extension into the complex plane, where, in a certain neighbourhood of the real axis, it has no poles [12, 13]. Consequently, the series (2.7) is a formal asymptotic solution of Eq. (1.2) outside $\bar{\Omega}$ for $k = \tau_\varepsilon$.

We will expand the eigenfunction $\psi_0(x)$ in a Taylor series at zero

$$\psi_0(x) = \psi_0(0) + O(r) \tag{3.3}$$

Rewriting (3.3) in terms of the variables $\xi = x\varepsilon^{-1}$, we obtained from the condition for series (2.5) and (2.6) to be matched,

$$v_0(\xi)\sim\psi_0(0), \quad \rho\to\infty \quad \xi_3\geqslant 0$$

There is a density $\mu(\xi)\in C^\infty(\omega)$ such that the potential of a simple layer $Y(\xi)$ with a given density is equal to unity on ω and belongs to $W_2^1(K\backslash\bar{\gamma})$ for any compactum $K\subset R^3$ [14]. We will put the function $v_0(\xi)$ equal to $\psi_0(0)(1-Y(\xi)/2)$ for $\xi_3\geqslant 0$ and $Y(\xi)/2$ for $\xi_3\leqslant 0$. By definition $v_0\in W_2^1(K\backslash\bar{\gamma})$ is a solution of the boundary-value problem (3.3) and, at infinity, has the asymptotic forms

$$v_0(\xi) = \begin{cases} \psi_0(0) - \tfrac{1}{2}c_\omega\psi_0(0)\rho^{-1} + O(\rho^{-2}), & \xi_3\geqslant 0 \\ \tfrac{1}{2}\psi_0(0)c_\omega\rho^{-1} + O(\rho^{-2}), & \xi_3\leqslant 0 \end{cases} \tag{3.4}$$

Hence, series (2.5) and (2.6) are matched at the first step.

Rewriting (3.4) in terms of the variables $x = \xi\varepsilon$ we obtain the principal terms of the asymptotic forms at zero for the coefficients of series (2.5) and (2.7). In particular,

$$\psi_1(x)\sim -\tfrac{1}{2}\psi_0(0)c_\omega r^{-1}, \quad P_0G(x, 0, \tau_1)\sim\tfrac{1}{2}\psi_0(0)c_\omega r^{-1} \tag{3.5}$$

It follows from the fact that the function $G(x, 0, k)$ is analytical with respect to the variable k in the neighbourhood of k_0 and its asymptotic form is analytical at zero with respect to the variable x that Eq. (2.7) holds for P_0.

We will now determine $\psi_1(x)$ and τ_1. A function $X(x)\in C^\infty(\bar{\Omega}\backslash 0)$ exists which is a solution of the boundary-value problem

$$(\Delta + k_0^2)X(x) = 2\pi\psi_0(0)\psi_0(x) \quad x\in\Omega, \quad \partial X/\partial\mathbf{n}=0 \quad x\in\Gamma_0\backslash 0 \tag{3.6}$$

and at zero has an asymptotic form $X(x) = r^{-1} + O(1)$. This assertion can easily be proved by putting

$$X(x) = r^{-1} \cos k_0 r + X_0(x)$$

where $X_0(x) \in C^\infty(\bar{\Omega})$ [9]. Hence it follows that the function $\psi_1(x) = -\frac{1}{2} c_\omega \psi_0(0) X(x)$ is a solution of boundary-value problems (3.1) for τ_1 from (2.4) and has a specified asymptotic form (3.5) at zero.

By extending the matching it is easy to construct the remaining asymptotically matched coefficient $\psi_j(x) = O(r^{-j})$ as $r \to 0$, $v_j(\xi) = O(\rho^j)$ for $\rho \to \infty$, $P_j(D_y) G(x, 0, k)$, which are solutions of the recurrent boundary-value problems in the corresponding regions, and the coefficients τ_j are found from the condition for the boundary problems (3.1) to be solvable for ψ_j.

We will show that (2.5) holds for $\operatorname{Im} \tau_2$. Suppose $B(R, t) = S(R) \backslash (\Omega \cup S(t))$. Integrating by parts for large R and real k we obtain

$$0 = \operatorname{Im} \int_{B(R,\, R^{-1})} \overline{G(x, 0, k)} \Delta G(x, 0, k)\, dx = k\sigma - \operatorname{Im} G(0, 0, k) + O(R^{-1})$$

Consequently, as $r \to 0$

$$\operatorname{Im} G(x, 0, k) = k\sigma + O(r)$$
$$\operatorname{Im} \varepsilon P_0 G(x, 0, k) = \varepsilon (P_0 k\sigma + O(r)) \tag{3.7}$$

Rewriting (3.7) in terms of the variables ξ we obtain that $\operatorname{Im} v_1(\xi) \sim P_0 k_0 \sigma$ as $\rho \to \infty$, $\xi_3 \leqslant 0$. The boundary-value problem for $\operatorname{Im} v_1$ has the form

$$\Delta \operatorname{Im} v_1 = 0 \quad \xi \notin \bar{\gamma}, \quad \partial \operatorname{Im} v_1 / \partial \xi_3 = 0 \quad \xi \in \gamma \tag{3.8}$$

The function

$$\operatorname{Im} v_1(\xi) = \psi_0^{-1}(0) P_0 k_0 \sigma v_0(\xi_1, \xi_2, -\xi_3)$$

is a solution of boundary-value problem (3.8) and as $\rho \to \infty$ has the asymptotic forms

$$\operatorname{Im} v_1(\xi) = \tfrac{1}{2} \pi \psi_0(0) k_0 \sigma c_\omega^2 \rho^{-1} + O(\rho^{-2}), \quad \xi_3 \geqslant 0$$
$$\operatorname{Im} v_1(\xi) = \pi \psi_0(0) k_0 \sigma c_\omega + O(\rho^{-1}), \quad \xi_3 \leqslant 0 \tag{3.9}$$

Rewriting (3.9) in terms of the variables x, we obtain that

$$\operatorname{Im} u_2 \sim \tfrac{1}{2} \pi \psi_0(0) k_0 \sigma c_\omega^2 r^{-1}, \quad r \to 0 \tag{3.10}$$

The boundary-value problem for $\operatorname{Im} u_2$, by virtue of (3.1), has the form

$$(\Delta + k_0^2) \operatorname{Im} u_2(x) = -2k_0 \operatorname{Im} \tau_2 \psi_0(x) \quad x \in \Omega, \quad \partial \operatorname{Im} u_2 / \partial \mathbf{n} = 0 \quad x \in \Gamma_0 \backslash 0 \tag{3.11}$$

We will put

$$\operatorname{Im} u_2 = \tfrac{1}{2} \pi \psi_0(0) k_0 \sigma c_\omega^2 X(x)$$

and $\operatorname{Im} \tau_2$ in accordance with (2.4). Then the function $\operatorname{Im} u_2(x)$ will be a solution of the boundary-value problem (3.11) and will have the asymptotic form (3.10) at zero.

Hence, we have constructed asymptotically matched series (2.4)–(2.7). It follows from estimates made in the paper listed in the previous footnote that the series (2.4)–(2.7) are asymptotic expansions of the pole τ_ε and of the corresponding quasi-eigenfunction $\psi_\varepsilon(x)$ at these norms.

The condition of the flattening of the boundary in the neighbourhood of the aperture affects the asymptotic form τ_ε. Suppose the boundary Γ_0 in the neighbourhood of the origin of coordinates is given by the equation

$$x_3 = F(x_1, x_2) \equiv a_1 x_1^2 + a_2 x_2^2 + O(r^3)$$
$$\omega_\varepsilon = \{x;\ x_3 = F(x_1, x_2),\ (x_1, x_2) \in \omega_\varepsilon\}, \quad \Gamma_\varepsilon = \Gamma_0 \backslash \bar{\omega}_\varepsilon'$$

In this case the asymptotic expansion of the pole τ_ε, generally speaking, contains powers of $\ln \varepsilon$, but mainly

$$\tau_\varepsilon = k_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + O(\varepsilon^3 \ln \varepsilon) \tag{3.12}$$

where the quantities τ_1 and $\operatorname{Im}\tau_2$ satisfy (2.4).

The absence of logarithms in the lowest forms of (3.12) is explained by the fact that the singular solution of boundary-value problem (3.6) and Green's function $G(x, 0, k)$ has the following asymptotic forms at zero [5, 15]:

$$X(x) \sim r^{-1} + {}^1/_4 (a_1 - a_2)(x_1^2 - x_2^2)(r + x_3)^{-2} - {}^1/_2 (a_1 + a_2) \ln (r + x_3)$$

$$2\pi G(x, 0, k) \sim r^{-1} - {}^1/_4 (a_1 - a_2)(x_1^2 - x_2^2)(r - x_3)^{-2} + {}^1/_2 (a_1 + a_2) \ln (r - x_3)$$

By (3.12), the representations (2.10) and (2.12) will hold for real $k = k(\varepsilon)$ and will have the asymptotic form (2.8) as $\varepsilon \to 0$.

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